Last Time: Ultimate Goal:

Compute a formula for

$$\binom{n}{k} := \#\{S \subseteq [n] : \#S = k\}$$

Have: $n! = \#\{\text{permutations of } [n]\}$

$$\frac{n!}{(n-k)!} \overset{*}{=} \#\{\text{ordered } k\text{-subsets of } [n]\}. \overset{*}{}$$

↳ Trick: Counting in Two Ways

Idea: If F_1 and F_2 are formulas for the cardinality of the same set, then $F_1 = F_2$.

Ex: When we proved $\frac{n!}{(n-k)!} = \#\{\text{ordered } k\text{-subsets of } [n]\}$, we actually counted permutations...

Prop: Let $0 \leq k \leq n$ be natural numbers. Then

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

"n choose k"

Pf: We count the set S of ordered k-subsets of $[n]$ in two ways. We already know the first count, namely $\#S = \frac{n!}{(n-k)!}$. For the second count, note every ordered k-subset is uniquely determined * by the following choices (i.e. counting procedure):

① Choose an unordered k-subset T of $[n]$ ($\binom{n}{k}$ options)

② Choose a permutation of T ($k!$ options).

Hence, by the Product principle, we have

$$\#S = \binom{n}{k} \cdot (k!). \quad \text{Hence } \frac{n!}{(n-k)!} = \#S = \binom{n}{k} \cdot (k!),$$

So we obtain $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ as desired. ☒

Counting Problems

Ex 1: How many different 3-letter initials are possible for English names? [26 26 26]

Sol: There are 26 English letters, i.e. $\#A = 26$ where $A = \{a, b, c, \dots, z\}$. We're asked to count the length 3 sequences in A. There are 26^3 such sequences. ☒

Ex 2: How many possible initials-sequences for English names have no two of the initials the same?

Sol 2a: We are asked to count ordered 3-subsets of $A = \{a, b, \dots, z\}$, so we have $\frac{26!}{(26-3)!}$ options ☒

Sol 2b: Every such initials sequence has the form $\rightarrow \underline{X}\,\underline{Y}\,\underline{Z}$ for $X, Y, Z \in A$ all distinct. We may enumerate these via the counting procedure below:

① Choose a first initial $X \in A$. (26 options)

② Choose a middle initial $Y \in A$ different from X. (26-1 options).

③ Choose a last initial $Z \in A$ different from X and Y. (26-2 options).

Hence, as this counting procedure constructs all elements of the set we want to count exactly once, we have $26 \cdot 25 \cdot 24$ elements in the set by the Product Principle. ☑

NB: DNA sequences have 4 bases: G C A T

Ex 3: How many length 5 DNA sequences ...

Ⓐ End w/ A?

Ⓑ Start w/ T and end w/ G?

Ⓒ Contain only A and T?

Ⓓ Do not contain C?

Sol 3A: All such sequences have the form: $\underline{b_1}\ \underline{b_2}\ \underline{b_3}\ \underline{b_4}\ \underline{A}$

Thus, these sequences are in one-to-one correspondence with the set of 4-sequences in $\{G, C, A, T\}$.

Thus there are $\#\{G, C, A, T\}^4 = 4^4$ such sequences. ☑

Sol 3B: All such sequences have form $\underline{T}\ \underline{b_1}\ \underline{b_2}\ \underline{b_3}\ \underline{G}$.

Thus, these sequences correspond w/ the set of 3-seqs in $\{G, C, A, T\}$, so there are 4^3 of them. ☑

Sol 3C: All such sequences have form $\underline{b_1}\ \underline{b_2}\ \underline{b_3}\ \underline{b_4}\ \underline{b_5}$

where $b_i \in \{A, T\}$. Thus we've asked to count the length 5 seqs in $\{A, T\}$; there are 2^5 such seqs. ☑

Sol 3D: All such sequences have the form $b_1\ b_2\ b_3\ b_4\ b_5$ where $b_i \in \{G, C, A, T\} \setminus \{C\} = \{G, A, T\}$. There are 3^5 such sequences.

Ex 4: How many positive integers at most 1000 ...

(A) Are divisible by 9?

(B) Have distinct digits?

(C) Have an odd number of digits?

Sol 4A: We are asked to count

$$S = \{n \in [1000] : n = 9k \text{ for some } k \in \mathbb{Z}\} \quad (*)$$

In particular, there is a one-to-one correspondence

$$S \longleftrightarrow T := \{k \in \mathbb{Z} : 9k \in [1000]\}$$

So $0 < k \leq \frac{1000}{9}$ iff $k \in T$.

i.e. $T = \{k \in \mathbb{Z} : 1 \leq k \leq \frac{1000}{9}\}$

$= \{k \in \mathbb{Z} : 1 \leq k \leq \frac{999}{9} = 111\} = [111]$

Hence $\#S = 111$.

NB: The above reasoning can be used to prove:

Prop: Let $n, d \in \mathbb{Z}_+$. Then $\#\{k \in [n] : d \mid k\} = \lfloor \frac{n}{d} \rfloor$ where $\lfloor \frac{n}{d} \rfloor = \max\{m \in \mathbb{Z} : m \leq \frac{n}{d}\}$.

Exercise: Prove the proposition!

Sol 4B: The set $S = \{n \in [1000] : n \text{ has distinct digits}\}$ is better expressed as $S = S_1 \cup S_2 \cup S_3 \cup S_4$ where $S_i := \{n \in S : n \text{ has } i \text{ digits}\}$. Note that

S_i is disjoint from S_j whenever $i \neq j$

(i.e. $\{S_1, S_2, S_3, S_4\}$ is pairwise disjoint).

Now we count each S_i separately:

Counting S_1: $S_1 = \{n \in [1000] : n$ has distinct digits and n has one digit$\}$.

Thus $S_1 = \{1, 2, 3, 4, \cdots, 9\}$, so $\#S_1 = 9$.

Counting S_2: We can express every elt of S_2 uniquely by

① Choose the first digit (9 options)
among $\{1, 2, \cdots, 9\}$

② Choose the second digit (9 options)
different to the first
from $\{0, 1, 2, \cdots, 9\}$

Hence $\#S_2 = 9 \cdot 9$ by the product principle.

Counting S_3: (left to you) ... so $\#S_3 = 9 \cdot 9 \cdot 8$.

Counting S_4: $S_4 \subseteq \{1000\}$, so $S_4 = \emptyset$.

Hence $\#S = \#S_1 + \#S_2 + \#S_3 + \#S_4$

$= 9 + 9 \cdot 9 + 9 \cdot 9 \cdot 8 + 0$ ☒

Sol 4C: Exercise! ☒

Anagrams.

Defn: Let W be a word in character set C.
An anagram of W is any word formed by rearranging the characters of W.

Ex: The word JIMMY has anagram IMJYM.

Non-Ex: MIMMJ is NOT an anagram of JIMMY because it ① doesn't use Y, ② uses too many M's

JMIY is NOT an anagram of JIMMY b/c it misses an M...

Ex 5: Enumerate the anagrams of JIMMY.

NB: The M's are indistinguishable... How do we cope with this fact?

Sol'n: Every anagram of JIMMY can be constructed uniquely by: $_ - ___$

① Choose the position of the J (5 options).

② Choose the positions of the M's ($\binom{4}{2}$ options).

③ Choose the positions of I (2 options).

④ Place the Y (1 option).

Hence by the product principle there are $5 \cdot \binom{4}{2} \cdot 2 \cdot 1$ such anagrams of JIMMY. ☒

Ex 6: Enumerate anagrams of the ANAGRAMMATICAL.

Sol'n: This word has: 5 A's, 2 M's, and one each of [N, G, R, T, I, C, and L]. So we count by:

①	Place the A's	$\binom{14}{5}$	⑧	Place C	2
②	Place the M's	$\binom{9}{2}$	⑨	Place L	1
③	Place N	7			
④	Place G	6			
⑤	Place R	5			
⑥	Place T	4			
⑦	Place I	3			

∴ There are $\binom{14}{5}\binom{9}{2} \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ anagrams of ANAGRAMMATICAL ☒